1 Introduction

In the literature fermion pairs are replaced by bosons in many known physical situations. This is normally performed with the help of boson mappings, that link the fermionic Hilbert space to another Hilbert space constructed with bosons. Of course boson mapping techniques are only useful when the Pauli Principle effects are somehow minimized. Historically boson expansion theories were introduced from two different points of view. The first one is the Beliaev - Zelevinsky - Marshalek (BZM) method [1], which focuses on the mapping of operators by requiring that the boson images satisfy the same commutation relations as the fermion operators. In principle, all important operators can be constructed from a set of basic operators whose commutation relations form an algebra. The mapping is achieved by preserving this algebra and mapping these basic operators. The Dyson mapping [2] is of the BZM type. It provides finite boson expansions but non - hermitian boson images. The boson expansions of the multipole operators contain only two-boson terms, while the expansions of the pair operators contain only one- and three- boson terms. In spite of being non- hermitian, boson images can be hermitized, if necessary. Hence, the non - hermiticity does not prevent us from using the Dyson mapping directly. However, to avoid this technical problem, in this paper we shall stick to the use of the quadrupole operator which generates hermitian images.

For the sake of completeness, let's make some comments on the second type of boson mappings mentioned above, the well known Marumori method [3]. It focuses on the mapping of state vectors. This method defines the operator in such a way that the matrix elements are conserved by the mapping and the importance of the commutation rules is left as a consequence of the requirement that matrix elements coincide in both spaces. The BZM and the Marumori expansions are equivalent at infinite order, which means that just with the proper mathematics one can go from one expansion to the other.

Quons are particles whose statistics interpolates between the boson and fermion statistics depending on a special deformation parameter which varies from +1 to -1 [4], [5]. This behaviour is easily observed by the q- deformed

commutation relations obeyed by the quons, which appears in the next section. Notice that there are subtle differences between the commutation relations of the usual quantum algebras and of the quons [6]. For q - bosons with zero angular momentum the commutation relations are identical, but this is not true for q-bosons with larger angular momentum.

In a recent work [7], we have shown that a deformed boson mapping of the Marumori type, when applied to a pairing interaction, provides excellent results, i.e., for a properly chosen deformation parameter the exact result is also achieved.

In this work we concentrate on the Dyson mapping [2] for a nuclear single $j = 5/2$ shell with identical particles. This problem is different from others in the literature [8] in what concerns the boson commutation relations used during the mapping. First of all, we briefly outline the main aspects of the mapping from a fermionic space to a quantum deformed bosonic space. Once the deformation parameter is set equal to one, the usual boson expansion is recovered. Then we give the expressions for the mapping of a quadrupole-quadrupole interaction, which is used for the comparison among the usual, the deformed mapping and the exact results obtained from a shell model calculation, in the same spirit as in refs. [9],[10].

2 The Generalized Dyson Boson Mapping

The fermionic pair (${A^\dagger}^J_M$) and multipole (B^J_M) operators for a single-j shell are defined as:

$$ {A^\dagger}^J_M = \frac{1}{\sqrt{2}} \sum_{m,m'} (j\ m\ j\ m'|J\ M)\ a^\dagger_m a^\dagger_{m'} \tag{1} $$

with $A^J_M = ({A^\dagger}^J_M)^\dagger$ and

$$ B^J_M = \sum_{m,m'} (j\ m\ j\ -m'|J\ M)\ (-1)^{j-m'} a^\dagger_m a_{m'} \quad , \tag{2} $$

where $a^\dagger_m$ ($a_{m'}$) are fermionic creation (annihilation) operators and the Clebsch-Gordan coefficients are defined as $(j\ m\ j\ m'|J\ M)$. It is straightforward to show that these operators satisfy the following commutation relations:

$$[B^{J_1}_{M_1}, B^{J_2}_{M_2}] = \sum_J (-1)^{2j+J}[1-(-1)^{J_1+J_2+J}]\sqrt{(2J_1+1)(2J_2+1)}$$

$$\times \ (J_1 M_1 J_2 M_2 | J M_1+M_2) \begin{Bmatrix} J_1 & J_2 & J \\ j & j & j \end{Bmatrix} B^J_{M_1+M_2}, \tag{3}$$

$$[A^{\dagger\,J_1}_{M_1}, A^{\dagger\,J_2}_{M_2}] = 0, \tag{4}$$

$$[A^{\dagger\,J_1}_{M_1}, A^{J_2}_{M_2}] = -\frac{1}{2}\delta_{J_1\ J_2}\delta_{M_1\ M_2}(1-(-1)^{2j-J_2})$$

$$-\frac{1}{2}\sum_J (-1)^{J+M_2}[(-1)^{J_1} - (-1)^{2j+J_2} - (-1)^{2j} - (-1)^{2j+J_1+J_2}]\sqrt{(2J_1+1)(2J_2+1)}$$

$$\times \ (J_1 M_1 J_2 - M_2 | J M_1 - M_2) \begin{Bmatrix} J_1 & J_2 & J \\ j & j & j \end{Bmatrix} B^J_{M_1-M_2}, \tag{5}$$

$$[B^{\dagger\,J_1}_{M_1}, A^{\dagger\,J_2}_{M_2}] = \sum_J (-1)^{J+M_1}[(-1)^{2j} - (-1)^{J_2}]\sqrt{(2J_1+1)(2J_2+1)}$$

$$\times \ (J_1 - M_1 J_2 M_2 | J M_2 - M_1) \begin{Bmatrix} J_1 & J_2 & J \\ j & j & j \end{Bmatrix} A^{\dagger\,J}_{M_2-M_1}. \tag{6}$$

The usual Dyson mapping consists in the bosonic expansion of the pair and multipole operators, where the pair operators include only one and three-boson terms and the multipole operators only two-boson terms. The coefficients of the bosonic expansions are determined in order to preserve the same commutation relations for the mapped operators. At this point we generalize the Dyson method, using q-deformed bosons in the Dyson bosonic expansion. The calculations in this work will be restricted to a $j = 5/2$ shell and will use only s,d and g bosons, which suffice to render the mapping exact.

Since we are interested in the quadrupole - quadrupole interaction, we shall only explain explicitly how the q- deformed quadrupole image is obtained. If we consider a single $j = 5/2$ shell, the commutation relations for this shell are found by substituting $j = 5/2$ into eqs. (3), (4), (5) and (6). The fermion multipole operators which appear in these commutation relations are going to be mapped onto functions of q- deformed boson operators as follows:

$$B^J_M \rightarrow Q^J_M.$$

We want an exact mapping. Thus, we are going to use one q-deformed boson for each possible pair operator and they satisfy the following commutation relations[5]:

$$[s, s^\dagger]_{q_s} = s\ s^\dagger - q_s\ s^\dagger\ s = 1, \tag{7}$$

$$[d_m, d^\dagger_{m'}]_{q_d} = d_m\ d^\dagger{}_{m'} - q_d\ d^\dagger{}_{m'}\ d_m = \ \delta_{m,m'}, \tag{8}$$

$$[g_m, g^\dagger{}_{m'}]_{q_g} = g_m\ g^\dagger{}_{m'} - q_g\ g^\dagger{}_{m'}\ g_m = \ \delta_{m,m'}, \tag{9}$$

In these equations the real numbers q_s,q_d and q_g are defined as the deformation parameters associated respectively with the s, d and g deformed bosons. Note that we recover the usual bosonic (fermionic) algebra when the three deformation parameters are equal to $+1(-1)$. The above commutation relations form an algebra called *quon algebra* and it has been used to study the small violations of Fermi or Bose statistics[4]. As usual s, d and g carry angular momentum 0, 2 and 4 and N_s, N_d and N_g are number operators associated with the s, d and g q-bosons.

Now we present the Dyson expansion of the multipole operators of interest :

$$\begin{aligned} Q^2_M &= A\ [(s^\dagger \otimes \tilde{d})^2_M + (d^\dagger \otimes s)^2_M] + B\ (d^\dagger \otimes \tilde{d})^2_M \\ &+C\ [(d^\dagger \otimes \tilde{g})^2_M + (g^\dagger \otimes \tilde{d})^2_M] + \ D\ (g^\dagger \otimes \tilde{g})^2_M \end{aligned} \tag{10}$$

$$Q^1_M = E\ (d^\dagger \otimes \tilde{d})^1_M\ +\ F(g^\dagger \otimes \tilde{g})^1_M \tag{11}$$

$$Q^3_M = G(d^\dagger \otimes \tilde{d})^3_M + H\ (g^\dagger \otimes \tilde{g})^3_M + \ J\ [(d^\dagger \otimes \tilde{g})^3_M + (g^\dagger \otimes \tilde{d})^3_M] \tag{12}$$

where we have used the notation for the spherical tensors :

$$\tilde{d}_{-m} = (-1)^m\ d_m \quad , \quad \tilde{g}_{-m} = (-1)^{-m} g_m$$

and

$$[T^{j_1} \otimes T^{j_2}]^J_M = \sum_{m,m'} (j_1\ m\ j_2\ -m'|J\ M)\ T^{j_1}_m\ T^{j_2}_{m'}.$$

The coefficients $A, B, C, D, E, F, G, H, J$ are determined imposing that Q^J_M satisfy the commutation relations obtained from the fermionic operators. Next we outline the steps followed in obtaining the coefficients of the multipole operators, bearing in mind that all commutation realtions used here come from eq. (3). From $[B^1_M, B^1_N]$ we have obtained E^2 and F^2, from $[B^2_M, B^2_N]$ we

have obtained A^2,B^2, C^2, D^2, BC and CD and from $[B^3_M, B^3_N]$ we have gotten GJ, HJ, J^2, H^2 and G^2. Of course many other commutation relations can be used to check the results of the coefficients once the number of commutation relations is larger than the amount of coefficients to be determined. We have finally obtained:

A= 0.8165	B=-0.5832	C= 0.9091
D= 0.8207	E= 0.7559	F= 1.8516
G=-0.9091	H=-0.3350	J=-1.235 .

Notice that these coefficients do not depend on q, although expressions (7),(8) and (9) have been used together with the Wick contraction technique for their calculation. They also agree with the results coming from the general formula for the usual Dyson mapping coefficients [11], [8]. Nonetheless, the quadrupole- quadrupole interaction does depend on q_s, q_d or q_g, as can be seen below.

From the expression of Q^2_M given in eq.(10) it follows that the Dyson mapped quadrupole-quadrupole operator is given by:

$$Q.Q = \sum_M (-1)^M \; Q^2_M \; Q^2_{-M} =$$

$$A^2 s^\dagger \; s + (A^2 + B^2 + C^2) d^\dagger \cdot \tilde{d}$$

$$+\frac{5}{9}(C^2 \;+\; D^2\;) g^\dagger \cdot \tilde{g}$$

$$+(A^2 \; s^\dagger \; s^\dagger \cdot \; \tilde{d}\tilde{d} \;+\; h.\; c.) + A^2(q_d + q_s)\; s^\dagger d^\dagger \cdot \tilde{d} s$$

$$+5 \; B^2 q_d \; \sum_J \begin{Bmatrix} 2 & 2 & 2 \\ 2 & 2 & J \end{Bmatrix} \; [d^\dagger \otimes d^\dagger]^J \cdot [\tilde{d} \otimes \tilde{d}]^J$$

$$+(5C^2 \sum_J \begin{Bmatrix} 4 & 2 & 2 \\ 2 & 4 & J \end{Bmatrix} \; [g^\dagger \otimes g^\dagger]^J \cdot [\tilde{d} \otimes \tilde{d}]^J + \; h.\; c.)$$

$$+5 \; (\; (q_d + q_g) C^2 \sum_J \begin{Bmatrix} 2 & 4 & 2 \\ 2 & 4 & J \end{Bmatrix} \; + 2BD \sum_J \begin{Bmatrix} 2 & 2 & 2 \\ 4 & 4 & J \end{Bmatrix} \;) \; [d^\dagger \otimes g^\dagger]^J \cdot [\tilde{d} \otimes \tilde{g}]^J$$

$$+5 \; D^2 q_g \; \sum_J \begin{Bmatrix} 4 & 4 & 2 \\ 4 & 4 & J \end{Bmatrix} \; [g^\dagger \otimes g^\dagger]^J \cdot [\tilde{g} \otimes \tilde{g}]^J$$

$$+(AB \; (q_d + 1) s^\dagger d^\dagger \cdot [\tilde{d} \otimes \tilde{d}]^2 \;+\; h.\; c.)$$

$$+(AC \; (q_d + 1) s^\dagger d^\dagger \cdot [\tilde{d} \otimes \tilde{g}]^2 \;+\; h.\; c.)$$

$$+(AC\ \frac{2\sqrt{5}}{3}\ s^\dagger g^\dagger \cdot [\tilde{d}\otimes\tilde{d}]^4\ +\ h.\ c.)$$

$$+(BC\ 5\ (1+(-1)^J\ q_d)\sum_J \begin{Bmatrix} 4\ 2\ 2 \\ 2\ 2\ J \end{Bmatrix}\ [d^\dagger\otimes g^\dagger]^J\cdot[\tilde{d}\otimes\tilde{d}]^J\ +\ h.\ c.)$$

$$(CD\ 5\ (1+(-1)^J\ q_g)\sum_J \begin{Bmatrix} 4\ 2\ 2 \\ 4\ 4\ J \end{Bmatrix}\ [g^\dagger\otimes d^\dagger]^J\cdot[\tilde{g}\otimes\tilde{g}]^J\ +\ h.\ c.)$$

$$+\,(AD\ \frac{2\sqrt{5}}{3}\ s^\dagger g^\dagger\cdot[\tilde{d}\otimes\tilde{g}]^4 +\ h.\ c.) \tag{13}$$

where Q^2_M is defined in eq.(10) and we have used the notation:

$$T^J\ \cdot\ U^J = (-1)^J\sqrt{2J+1}\ [T^J\otimes U^J].$$

Thus, we have obtained the q-deformed Dyson mapping of the quadrupole - quadrupole operator.

3 Applications and Results

In order to study the properties of the q-deformed Dyson expansion we propose to compare the diagonalization of a quadrupole hamiltonian mapped onto the bosonic subspace generated by s, d and g bosons with the exact shell model calculation restricted to four fermions in a $j = 5/2$ shell. To achieve our goal, one has to introduce a basis for the diagonalization of the desired hamiltonian, which consists of two q- deformed boson states given by

$$|ss>=\frac{1}{\sqrt{(1+q_s)}}s^\dagger\ s^\dagger\ |0> \tag{14}$$

$$|dd>=\frac{1}{\sqrt{(1+(-1)^\lambda\ q_d)}}\sum_{m,m'}(2\ m\ 2\ m'\ |\lambda\ m+m')\ d^\dagger_m\ d^\dagger_{m'}\ |0> \tag{15}$$

$$|gg>=\frac{1}{\sqrt{(1+(-1)^\lambda\ q_g)}}\sum_{m,m'}(4\ m\ 4\ m'\ |\lambda\ m+m')\ g^\dagger_m\ g^\dagger_{m'}\ |0> \tag{16}$$

$$|sd>=s^\dagger\ d^\dagger_m\ |0> \tag{17}$$

$$|sg>=s^\dagger\ g^\dagger_m\ |0> \tag{18}$$

$$|dg>=\sum_{m,m'}(2\ m\ 4\ m'\ |\lambda\ m+m')d^\dagger_m\ g^\dagger_{m'}\ |0> \tag{19}$$

where λ is the total angular momentum of the basis state.

The main results of our diagonalization can be summarized in Fig. I. There we make an analysis of the behavior of the q- deformed bosonic spectrum compared with the one obtained through the exact shell-model diagonalization procedure for four fermions in a single $j = 5/2$ shell. In this simple case, we get just three states for the fermionic spectrum with angular momenta 0,2 and 4. The bosonic results however are highly dependent on the values of the q- parameters. We start by comparing the spectrum obtained making $q_s = 1, q_d = 1$ and $q_g = 1$, i.e., turning off the effects of the deformation. In this case the agreement is very poor. Also, note that the spurious part of the bosonic spectrum lies well above the "physical" states.

Instead of doing just a fit to the exact spectrum in order to find out the best values for the q parameters, we have decided first to try to find a criterion for fixing those parameters. In this work we are considering a relatively low value for the shell angular momenta j, which is known not to be a very favorable situation for the quadrupole- quadrupole boson expansion. However, it is our intention to test our deformed bosonic expansion even for that not so favorable case and show that our description can improve considerably the results. We start by considering again the pair operator $A^{\dagger\ J}_M$ defined in equation (1) and identifying it with the boson deformed operators s, d and g respectively for $J = 0, 2$ and 4 in analogy with reference [12]. Of course, the pair operator cannot be treated as a real boson operator and this can be made clear, for example, when we take the norm of the state:

$$\frac{1}{\sqrt{2}}[A^{\dagger\ J} \otimes A^{\dagger\ J}]^{\lambda}_{\mu}|0>= \frac{1}{\sqrt{2}} \sum_{m,m'} (JmJm'|\lambda\mu) A^{\dagger\ J}_{m} A^{\dagger\ J}_{m'}|0> .$$

Note that the state considered in the above equation has, according to our assumption, the bosonic images defined in equations (14), (15) and (16). Those are the basis states that define the ground state of the system taking $\lambda = 0$. On the other hand one may easily show that:

$$\frac{1}{2} < 0|[A^{J} \otimes A^{J}]^{0}_{0}[A^{\dagger\ J} \otimes A^{\dagger\ J}]^{0}_{0}|0>= 1 - 2(2J+1)^2 \left\{ \begin{array}{ccc} j & j & J \\ j & j & J \\ j & j & 0 \end{array} \right\}. \qquad (20)$$

The term depending on the 9-J simbol expresses the effect of the Pauli principle and its magnitude might be used to estimate the validity of our bosonic identification between the pair and deformed boson operators. Considering now the norm of the two-boson states:

$$\frac{1}{2} < 0|sss^{\dagger}s^{\dagger}|0 >= 1 + \frac{q_s - 1}{2} \tag{21}$$

$$\frac{1}{2} < 0|[d \otimes d]^0_0 [d^{\dagger} \otimes d^{\dagger}]^0_0|0 >= 1 + \frac{q_d - 1}{2} \tag{22}$$

$$\frac{1}{2} < 0|[g \otimes g]^0_0 [g^{\dagger} \otimes g^{\dagger}]^0_0|0 >= 1 + \frac{q_g - 1}{2} \tag{23}$$

and comparing the three above equations with equation (20) we may get an estimation for the parameters q_s, q_d and q_g. For $j = 5/2$ we obtain $q_s = 0.3$ and $q_d = 0.6$. For q_g however we do not find a reasonable value (between -1 and 1). This seems to be a problem inherent to a small j-shell value, once we check that for bigger values of j we allways end up with reasonable values for the three q parameters (including j=7/2). So we decide to keep $q_g = 1$. Observing again Fig. I, one can see that our choice for the parameters improves considerably the agreement with the exact shell model calculation.

Finally, turning back to the simple fitting procedure, we search for the best matching between fermionic and bosonic levels varying two deformation parameters, while keeping the third one fixed . We find that the best agreement is reached when we fix the g-boson deformation parameter ($q_g = 1$), assigning to q_s and q_d the values 0.2 and 0.4 respectively, as also shown in Fig. I. Of course we also might vary independently the three parameters. However, we have checked numerically that the quality of the fitting is not considerably improved in that case.

In summary, in this work we find the deformed bosonic image of the quadrupole-quadrupole hamiltonian through the implementation of a q - deformed Dyson mapping based in the so-called *quon* algebra. Explicit results are presented for a system of four identical fermions in a single $j = 5/2$ shell and the bosonic expansion is written in terms of deformed s, d and g bosons. We then try to find a way to fix the q parameters associated with each boson

operator by their identification with the pair operator carrying the corresponding angular momenta. Compared with the usual non-deformed result for the energy spectrum, a remarkable improvement is achieved using this method. However, for the g boson we do not find a reasonable value for the deformation parameter in the case considered here. As for larger shells and the same number of fermions we can find reasonable q values for the three parameters using that very same method, we believe that it is worthwhile to proceed! further this investigation in a

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4 Figure Captions

FIGURE I. Results for the energy levels for a system composed of four particles in a j=5/2 shell interacting through the quadrupole-quadrupole force in a) the exact shell-model diagonalization, b)using the deformed Dyson boson expansion of this work with $q_s = 1.0, q_d = 1.0$ and $q_g = 1.0$, c)using $q_s = 0.3, q_d = 0.6$ and $q_g = 1.0$ (physical interpretation) and d)$q_s = 0.2, q_d = 0.4$ and $q_g = 1.0$ (best fitting). See the text for details.

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